Multiplication of Matrices :- Two matrices are said to be conformable (or compatible) for multiplication $\Leftrightarrow$ the no. of columns in the first matrix is equal to the no. of rows of the second matrix.

eg :- $A = \begin{bmatrix} 2 & 3 & -2 \\ 4 & 5 & 2 \\ 2 & 1 & 3 \end{bmatrix}_{3\times 3}$ & $B = \begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}_{3\times 2}$

$\Rightarrow$ **Transpose of a matrix** :- A matrix obtained by changing the rows of a given matrix 'A' into columns is called transpose of 'A'. It is denoted by 'A^T' (or) 'A''.

eg :- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

$\Rightarrow$ $\boxed{\text{Note}}$: (i). $(A^T)^T = A$

(ii). $(A+B)^T = A^T + B^T$

(iii). $(A-B)^T = A^T - B^T$

(iv). $(AB)^T = B^T A^T$

(v). $(KA)^T = KA^T$, where 'K' is a scalar.

$\Rightarrow$ **Symmetric Matrix** :- A square matrix 'A' is said to be symmetric matrix if $\boxed{A^T = A}$

eg :- $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

⇒ **Skew - Symmetric Matrix** :- A square matrix 'A' is said to be skew-symmetric matrix if $\boxed{A^T = -A}$

eg:- $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

⇒ **Determinant of a square matrix** :- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the number "ad-bc" is called "determinant of a matrix" 'A' of order 2x2. It is denoted by '$|A|$' (or) '$\det A$' (or) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\therefore |A| = ad - bc$.

eg:- If $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ then $|A| = 12 - 10 = 2$.

⇒ **Singular matrix** :- If the determinant of a square matrix is zero, then it is called "a singular matrix".

$\therefore \boxed{\det A = 0}$

eg:- If $A = \begin{bmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then

$$|A| = -3 - 2(-6) + 3(32 - 35)$$
$$= -3 + 12 - 9$$
$$= 12 - 12$$
$$= 0$$

⇒ **Non - singular matrix** :- If the determinant of a square matrix is not equal to zero, then it is called "non-singular matrix".

$\therefore \boxed{\det A \neq 0}$ eg:- $A = \begin{bmatrix} 4 & 7 \\ 5 & 3 \end{bmatrix}$

Note :- 1). If $A = \begin{bmatrix} 4 & -2 \\ x & 3 \end{bmatrix}$ is a singular matrix then find 'x'.

Given

$\therefore \quad |A| = 0$

$\Rightarrow \begin{vmatrix} 4 & -2 \\ x & 3 \end{vmatrix} = 0 \Rightarrow 12 + 2x = 0 \Rightarrow 2x = -12 \Rightarrow \boxed{x = -6}$

2). If $A = \begin{bmatrix} 2 & -1 & 4 \\ x & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ is a singular matrix then find 'x'.

$\therefore$ Given $|A| = 0$

$$\Rightarrow \begin{vmatrix} \overset{+}{2} & \overset{-}{-1} & \overset{+}{4} \\ x & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 0$$

$\Rightarrow 2(-4) + 1(-2) + 4(2x) = 0$

$\Rightarrow -8 - 2 + 8x = 0 \Rightarrow 8x = 10 \Rightarrow \boxed{x = 5/4}$

3). If 'A' is a square matrix such that $\boxed{A^2 = A}$ then 'A' is called "idempotent."

4). If 'A' is a square matrix such that $\boxed{A^2 = I}$ then 'A' is called "involutory".

5). If 'A' is a square matrix such that $\boxed{A^m = O}$, where 'm' is a positive Integer, then 'A' is called "nilpotent."

If 'm' is a least +ve integer such that $\boxed{A^m = O}$ then 'A' is called nilpotent of index 'm'.

→ The conjugate of a matrix :- The matrix obtained from any given matrix 'A', on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of 'A'. It is denoted by '$\bar{A}$'.

Eg: $A = \begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2\times 3}$ then $\bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2\times 3}$

Note:
(i). $\overline{(\bar{A})} = A$

(ii). $\overline{(A+B)} = \bar{A}+\bar{B}$

(iii). $\overline{KA} = \bar{K}\bar{A}$, 'k' being any complex number.

(iv). $\overline{(AB)} = \bar{A}\cdot\bar{B}$, 'A' & 'B' being conformable for multiplication

→ Conjugate Transpose of a matrix :- The transpose of conjugate matrix is called the "conjugate transpose of a matrix." It is denoted by 'A^{θ}'.

i.e, $\boxed{A^{\theta} = (\bar{A})^{T}}$ (or) $\boxed{A^{\theta} = \overline{(A^{T})}}$ & $(A^{\theta})^{\theta} = A$

Eg:-

If $A = \begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2\times 3}$; $A^{\theta} = (\bar{A})^{T}$

$= \begin{bmatrix} 5 & 3+i & 2i \\ 0 & 1-i & 4+i \end{bmatrix}_{2\times 3}$

$= \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3\times 2}$

⇒ **Orthogonal matrix** :– A square matrix 'A' is said to be orthogonal, if $\boxed{AA^T = A^TA = I}$. i.e, $\boxed{A^T = A^{-1}}$

⇒ **Adjoint of a square matrix** :– Let 'A' be a square matrix of order 'n'. The transpose of the matrix got from 'A' by replacing the elements of 'A' by the corresponding co-factors is called the adjoint of 'A' and it is denoted by "adjA".

i.e, $\text{adj}A = (\text{Co-factor matrix})^T$.

⇒ **Inverse of a matrix** :– Let 'A' be any square matrix ∃ a matrix 'B' ∋ $\boxed{AB = BA = I}$ then 'B' is called inverse of 'A'. It is denoted by 'A^{-1}'.

i.e, $\boxed{A^{-1} = \dfrac{\text{adj}A}{|A|}}$ ∵ $|A| \neq 0$.

Eg:- If $A = \begin{bmatrix} \overset{(+)}{2} & \overset{(-)}{3} & \overset{(+)}{4} \\ \overset{(-)}{4} & \overset{(+)}{3} & \overset{(-)}{1} \\ \overset{(+)}{1} & \overset{(-)}{2} & \overset{(+)}{4} \end{bmatrix}$ then find adj A & A^{-1} ?

Co-factor matrix :– $\begin{bmatrix} +10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & +14 & -6 \end{bmatrix}$

∴ $\text{Adj}A = (\text{Co-factor matrix})^T = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & +14 \\ 5 & -1 & -6 \end{bmatrix}$

& $\boxed{A^{-1} = \dfrac{\text{Adj}A}{|A|}}$; $|A| \neq 0$.

Here $|A| = 2(12-2) - 3(16-1) + 4(8-3)$

$= 20 - 45 + 20 = 40 - 45 = -5 \neq 0.$

$$\therefore \bar{A}^{1} = \frac{1}{-5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}.$$

⇒ Hermitian matrix :- A square matrix 'A' such that $\boxed{A^T = \bar{A}}$ (or) $\boxed{(\bar{A})^T = A}$ is called a "Hermitian matrix".

this can also be written as $\boxed{\overline{(A^T)} = A}$.

eg:- $A = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

& $\bar{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$\therefore$ Here $\boxed{A^T = \bar{A}}$

$\therefore$ 'A' is Hermitian matrix".

Note : the elements of the principal diagonal of a Hermitian matrix must be real.

⇒ skew-Hermitian matrix :- A square matrix 'A' such that $\boxed{A^T = -\bar{A}}$ (or) $\boxed{A = -(\bar{A})^T}$ is called "skew-Hermitian matrix

they can also be written as $\boxed{\overline{(A^T)} = -A.}$

Eg:- $\boxed{A = \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}}$ then $A^T = \begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix} = -\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$

Here

$$= -\bar{A}$$

$$\boxed{\bar{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}}$$

$$\boxed{\therefore A^T = -\bar{A}}$$

∴ 'A' is Skew-Hermitian matrix.

Note: the elements of the principal diagonal of a skew-Hermitian matrix must be all zero (or) purely imaginary.

⇒ unitary matrix :- A square matrix 'A' such that

$$\boxed{(\bar{A})^T = A^{-1}}$$

i.e., $(\bar{A})^T A = A(\bar{A})^T = I$

(or)

$A^\theta A = AA^\theta = I$ is called a "unitary matrix".

Eg:- $A = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$

then $\bar{A} = \begin{bmatrix} -\frac{1}{2}i & 1/2\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}i \end{bmatrix}$

LHS

then $= (\bar{A})^T = \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2\sqrt{3}} & \cancel{-\frac{1}{2}i} \end{bmatrix}$ ——— (1)

R.H.S = $\boxed{A^{-1} = \frac{adj A}{|A|}}$ [∵ $|A| \neq 0$]

$$A^{-1} = \frac{\begin{bmatrix} \frac{1}{2}i & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}}{-\frac{1}{4} - \frac{3}{4}} = \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}i \end{bmatrix} \quad \text{--- (2)}$$

from (1) & (2)

(1) = (2).

∴ $\boxed{(\bar{A})^T = A^{-1}}$

∴ 'A' is a unitary matrix.

Eg:- Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

L.H.S = $(\bar{A})^T = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$ --- (1)

R.H.S = $A^{-1} = \frac{adj A}{|A|} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$

$\left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) - \left(\frac{-1+i}{2}\right)\left(\frac{1+i}{2}\right)$

simplification

$A^{-1} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \Big/ \frac{\cancel{4}}{\cancel{4}}$

$A^{-1} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$ --- (2)

from (1) & (2)

$\boxed{(\bar{A})^T = A^{-1}}$ ∴ 'A' is unitary.

theorem :- Every Square matrix can be expressed as the Sum of Symmetric and skew-symmetric matrices in one and only way (uniquely).

(or)

S.T any square matrix $A = B + C$, where B = Symmetric matrix.
C = Skew-Symmetric matrix.

Proof :- Let 'A' be any square matrix. We can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$A = P + Q \text{ (say)}$$

Here $P = \frac{1}{2}(A + A^T)$; $Q = \frac{1}{2}(A - A^T)$

$P^T = \left[\frac{1}{2}(A + A^T)\right]^T$; $Q^T = \left[\frac{1}{2}(A - A^T)\right]^T$

$= \frac{1}{2}(A^T + A)$; $= \frac{1}{2}(A^T - A)$

i.e, $\boxed{P^T = P}$; $= -\frac{1}{2}(A - A^T)$

i.e, $\boxed{Q^T = -Q}$

$\therefore$ square matrix (A) = Symmetric matrix (P) + Skew symmetric matrix (Q)

$$\boxed{A = P + Q}.$$

uniqueness :- If possible let $\boxed{A = R + S}$, & $A^T = (R + S)^T = R^T + S^T = R - S$

where R = Symmetric matrix $(R^T = R)$
S = Skew Symmetric matrix $(S^T = -S)$

Now $\frac{1}{2}(A+A^{T}) = \frac{1}{2}[R+S+R-S] = R.$

$\frac{1}{2}(A-A^{T}) = \frac{1}{2}[R+S-R+S] = S.$

∴ The representation is unique.

Eg:- Express the matrix 'A' as a sum of symmetric and skew-symmetric.

$$A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$$

We have to prove $A = P+Q$, where $P^{T}=P$ & $Q^{T}=-Q$.

where $P = \frac{1}{2}(A+A^{T})$

$Q = \frac{1}{2}(A-A^{T})$

Now check $P = \frac{1}{2}(A+A^{T}) \Rightarrow P = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$ — (1)

then $P^{T} = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$

∴ $\boxed{P^{T}=P}$ (∵(1))

$Q = \frac{1}{2}(A-A^{T}) \Rightarrow Q = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$ — (2)

then $Q^{T} = \frac{1}{2}\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix} \Rightarrow Q^{T} = -\frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} \Rightarrow \boxed{Q^{T}=-Q}$

∴ $\boxed{A = P+Q}$

1. S.T $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol:- Given $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Consider $AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta+\sin^2\theta & -\cos\theta\sin\theta+\cos\theta\sin\theta \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & \cos^2\theta+\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I \qquad \boxed{\therefore AA^T = I}$$

$\therefore$ 'A' is orthogonal.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol:- Given $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A \cdot A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \frac{\cancel{9}}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore AA^T = I$

$\|^{ly}$ $A^T A = I$

$\therefore$ 'A' is orthogonal matrix.

③. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal. (16)

Sol^n:- For orthogonal matrix $AA^T = I$

so, $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{—— (1)}$$

by solving $2b^2-c^2=0$; $a^2-b^2-c^2=0$

$\Rightarrow 2b^2 = c^2$; $a^2-b^2-2b^2=0$

$\Rightarrow \boxed{c = \pm\sqrt{2}\,b}$; $\boxed{a^2 = 3b^2}$

$\Rightarrow a = \pm\sqrt{3}\,b.$

from the diagonal elements of Eq^n (1)

$4b^2+c^2 = 1$

$4b^2+2b^2 = 1$

$6b^2 = 1$

$\boxed{b = \pm 1/\sqrt{6}}$ & $c = \pm\dfrac{\sqrt{2}}{\sqrt{6}}$

$\boxed{c = \pm 1/\sqrt{3}}$.

& $a = \pm\dfrac{\sqrt{3}}{\sqrt{6}}$

$\boxed{a = \pm 1/\sqrt{2}}$

Note :- 1). The Inverse of a non-singular Symmetric matrix is symmetric.

2). If 'A' is a symmetric matrix, then prove that 'adj A' is also symmetric.

3). Matrix multiplication is associative

i.e., If A, B, C are matrices then $(AB)C = A(BC)$

4). Multiplication of matrices is distributive with respect to addition of matrices.

i.e., $A(B+C) = AB+AC$

$(B+C)A = BA+CA$.

5). If 'A' is a matrix of order $m \times n$ then $AI_n = I_n A = A$

& $I^n = I$ & $O^n = O$.

6). If A, B are orthogonal matrices, each of order 'n' then 'AB' and 'BA' are orthogonal matrices.

7). The Inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

1. Solve the Eq^{n}s $3x+4y+5z=18$, $2x-y+8z=13$ & $5x-2y+7z=20$ by using matrix Inversion method. (18).

Sol^n:- the given eq^{n}s in matrix form is $AX=B$

$$\Rightarrow \boxed{X = A^{-1}B} \quad \text{—— (1)}$$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

We have to find A^{-1} :-

$$\boxed{A^{-1} = \frac{\text{adj}A}{|A|}} \quad \because |A| \neq 0. \quad \text{—— (2)}$$

Now $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$

$$= 3(-7+16) - 4(14-40) + 5(-4+5) = 136 \neq 0.$$

Now $\text{adj}A = \left(\begin{matrix}\text{Co-factor matrix} \\ \text{of 'A'}\end{matrix}\right)^T$

$$= \begin{bmatrix} (-7+16) & -(14-40) & (-4+5) \\ -(28+10) & (21-25) & -(-6-20) \\ (32+5) & -(24-10) & (-3-8) \end{bmatrix}^T = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}^T$$

$$= \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

from (2) $\Rightarrow$ $A^{-1} = \frac{adj A}{|A|}$

$$= \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

from (1) $\Rightarrow$ $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162 - 494 - 740 \\ 468 - 52 - 280 \\ 18 + 338 - 220 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x = 3$; $y = 1$; $z = 1$ //

<u>Submatrix</u> :- Any matrix obtained by deleting some rows (or) columns (or) both of a given matrix is called <u>sub-matrix</u>.

<u>Eg</u>:- Let $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}_{3\times4}$ then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2\times3}$ is a sub-matrix of 'A' obtained by deleting 1^{st} row and 4^{th} column of 'A'.

Minor of a matrix :- Let 'A' be an $m \times n$ matrix. the determinant of a square sub-matrix of 'A' is called a "minor of the matrix".

Note : If the order of the square sub-matrix is 't' then its determinant is called a minor of order 't'.

Eg:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4 \times 3}$ be a matrix.

$\Rightarrow$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'.

$\rightarrow$ $|B| = (2-3) = -1$ is a minor of order '2'.

$\Rightarrow$ $C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'.

$$\rightarrow |C| = 2(7-12) - 1(21-10) + 1(18-5)$$
$$= 2(-5) - 1(11) + 1(13)$$
$$= -10 - 11 + 13$$
$$= 13 - 21$$
$$= -8$$ is a minor of order '3'.

Rank of a matrix :— Let 'A' be a rectangular matrix of order $m \times n$, submatrix of a matrix 'A' is any matrix obtained from 'A' by omitting some rows and columns in 'A'.

Rank of a matrix 'A' is the +ve 'integer' 'r' such that ∃ atleast one r-rowed square matrix with non-vanishing determinant while every $(r+1)$ (or) more rowed matrices have vanishing determinants.

"The rank of a matrix is the largest order of a non zero minor of matrix."

Rank of 'A' is denoted by $r(A)$ (or) $\rho(A)$

Note : (1). Rank of 'A' & 'A^T' is same.

(2) Rank of null-matrix is zero.

(3) For a rectangular matrix 'A' of order $m \times n$
rank of 'A' $\leq \min(m,n)$

(4). Rank of Identity matrix 'I_n' is 'n'.

(5). If 'A' is a matrix of order 'n' and 'A' is non-singular [i.e. $|A| \neq 0$] then $\rho(A) = n$.

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1. find the Rank of the matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3\times3}$.

Sol:- $|A| = -1(18-1) + 0 + 6(3+30)$

$= -17 + 198 \neq 0.$

$\therefore |A| \neq 0.$

from Note number (5).

$\therefore \boxed{\rho(A) = 3}$

2) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}_{3\times3}$.

Sol:- $|A| = 1(24-25) - 2(18-20) + 3(15-16)$

$= -1 + 4 - 3$

$= 0.$

$\therefore \rho(A) < 3.$

So, consider a minor of order '2' $= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$

$\therefore$ Hence there is atleast a minor of order '2', which is not zero.

$\therefore \boxed{\rho(A) = 2}$ [$\because$ Note (5)]

3). Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times 4}$.

Sol:- Here the matrix is of order 3×4.

$\therefore$ It is a rectangular matrix.

from Note (3)

Its rank $\leq \min(3,4) = 3$.

$\therefore$ Highest order of the minor will be '3'.

Let us consider the minor $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}_{3\times3}$.

Here $|A| = 1(-49) - 2(-56) + 3(35-48)$

$= 24 \neq 0$.

$\therefore$ from Note (5), $\boxed{\rho(A) = 3}$.

Elementary transformation (or operations) on a matrix :—

(i). Interchange of two rows : If i^{th} row and j^{th} row are interchanged, it is denoted by $R_i \leftrightarrow R_j$

(ii). Multiplication of each element of a row with a non-zero scalar. If i^{th} row is multiplied with 'k' then it is denoted by $R_i \leftrightarrow kR_i$.

iii). multiplying every element of a row with a non-zero scalar and adding to the corresponding elements of another row.

If all the elements of i^{th} row are multiplied with 'K' and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + KR_i$

$\|^{ly}$ we can write the column operations instead of 'R' write the 'C'.

<u>Equivalance of matrices</u> :— If 'B' is obtained from 'A' after a finite chain of elementary transformation then 'B' is said to be equivalent to 'A'. It is denoted by $B \sim A$.

$\boxed{\text{Note}}$: If 'A' and 'B' are two equivalent matrices, then

rank A = rank B.

If two matrices 'A' and 'B' have the same size and the same rank, then the two matrices 'A' and 'B' are Equivalent.

<u>Zero row</u> :— If all the elements in a row of a matrix are zeros, then it is called "<u>Zero row</u>".

<u>Non-zero row</u> :— If there is atleast one non-zero element in a row, then it is called "non-zero <u>row</u>."

Echelon form of a matrix :-

A matrix is said to be in Echelon form, if

(i). Zero rows, If any exists, they should be below the non-zero row.

(ii). the first non-zero entry in each non-zero row is equal to '1'.

(iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: 1). the number of non-zero rows in echelon form of 'A' is the rank of 'A'.

2). the rank of the transpose of a matrix is the same as that of original matrix.

3). condition (ii) is optional.

Eg:- (1).
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
is row echelon form.

(2).
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
is a row echelon form.

(26)

Q. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelon form & hence find its rank.

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Applying row operations on 'A'.

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$; $R_4 = R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 = R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 = R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ this is an Echelon form

rank = no. of non-zero rows

= 3 //

$\boxed{\therefore \rho(A) = 3}$

(2). Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing into Echelon form.

Sol:- Given $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on 'A'.

$R_1 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$; $R_3 = R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_2 = \frac{R_2}{7}$, $R_3 = \frac{R_3}{9}$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ this is the Echelon form of 'A'

rank = no. of non-zero rows

$\boxed{\rho(A) = 2}$

(3). Reduce the matrix to Echelon form & find its rank of $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ (28).

Applying row operations on 'A'

$R_2 = R_2 + R_1$; $R_3 = R_3 + 2R_1$; $R_4 = R_4 - R_1$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 = 2R_3 - 11R_2$; $R_4 = R_4 + 2R_2$

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 = 6R_4 + R_3$

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ this is an Echelon form

$$\boxed{\rho(A) = 4}.$$

4. $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ Find the rank of the matrix.

Sol:- $R_2 = R_2 - 2R_1$; $R_3 = R_3 - 4R_1$; $R_4 = R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_4 = R_4 - 3R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the matrix is in Echelon form.

$\rho(A)$ = no. of non-zero rows

$\boxed{\rho(A) = 2}$

(5). For what value of 'k' the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'. (30).

Sol: Let $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

$R_2 = 4R_2 - R_1$; $R_3 = 4R_3 - kR_1$; $R_4 = 4R_4 - 9R_1$ we get

$$A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

the given matrix is of the order 4×4.

If its rank is '3' (given), then we must have $|A| = 0$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \Rightarrow \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0 \end{array}$$

$$\Rightarrow \left[3(8-4k)\right] - 1\left[(8-4k)(4k+27)\right] = 0$$

$$\Rightarrow (8-4k)\left[3 - 4k - 27\right] = 0$$

$$\Rightarrow (8-4k)(-24-4k) = 0$$

$\Rightarrow 8-4k = 0$ $\boxed{k=2}$ | $-24-4k=0$ $\Rightarrow \boxed{k=6}$.

6. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ find the rank of $A, B, A+B, AB$ & BA.

7. Find the rank of

(i). $$A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

(ii). $$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

(iii). $$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

(iv). $$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(v) $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$$

(vi). $$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$$

(vii). $$\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

8. For the value of 'k' such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is '2'.

Solⁿ:- Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given rank of $A = \rho(A) = 2$

$\therefore |A| = 0$

$\Rightarrow \boxed{k = 4}$

(or) $R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & k-4 & 7-6 \\ 0 & 0 & 1 \end{bmatrix}$$

The given matrix is of the order 3×3. If its rank is '2' (given), then we must have $|A| = 0$.

$$\rightarrow \begin{vmatrix} k-4 & 7-6 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow (k-4) - 0 = 0$$

$\boxed{k = 4}$

9. Find the value of 'k' such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is '2'.

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is '2'

10. Find the value of 'k' if the Rank of the matrix 'A' is '2' where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$$

Solⁿ:- $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$; $R_4 = R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Given $\rho(A) = 2$, we must have $|A| = 0$

$$\begin{vmatrix} -3 & -1 \\ k-1 & -1 \end{vmatrix} = 0 \Rightarrow -(k-1) - 3 = 0 \Rightarrow \boxed{k = -2}$$

Reduction to Normal form :- It is another Important method of finding rank of a matrix.

—* Every $m\times n$ matrix of rank 'r' can be reduced to the form I_r, (or) $[I_r, 0]$ (or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row (or) column operations, where 'I_r' is the r-rowed unit matrix." The above form is called "normal form" (or) "1st canonical form" of a matrix.

corollary (1) :- The rank of a $m\times n$ matrix 'A' is 'r' $\Leftrightarrow$ if it can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row and column operations.

—: Problems :—

(1). $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and hence find the rank.

Sol:- Applying $C_1 \leftrightarrow C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_2 = \frac{R_2}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 = C_3 - 2C_1$; $C_4 = C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = \frac{C_2}{2}$; $C_4 = \frac{C_4}{3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 = C_3 - C_2$; $C_4 = C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

this is in the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ (normal form)

$\therefore$ the number of non-zero rows $= 2$

$\therefore \boxed{\rho(A) = 2}$

(2). $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing into canonical form.

Sol:-

$R_1 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_2 = R_2 - 4R_1$
$R_3 = R_3 - 2R_1$
$R_4 = R_4 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = R_2/2$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 = 3R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = C_2 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = C_2/3$; $C_3 = C_3/3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 + 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4/(-8)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ this is in the form of $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ (normal form)

∴ The no. of non-zero rows = 3

$\boxed{\therefore \rho(A) = 3}$

(3). By reducing the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ into normal form, find its rank.

Sol:- $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$R_4 = R_4 - 6R_1$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

$R_3 = 5R_3 - 4R_2$; $R_4 = 5R_4 - 9R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$

$R_3 = R_3/11$

$R_4 = R_4/11$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix}$

$R_4 = R_4 - R_3$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 = C_2 + C_1$

$\sim \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 = C_3 + 2C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_4 = C_4 + 4C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 = \frac{C_2}{5}$; $C_3 = \frac{C_3}{3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 = C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 - 7C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = \frac{C_4}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

this is of the form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$. Hence Rank of 'A' is '3'.

4. Reduce the matrix 'A' to normal form and hence find its rank. $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ (3)

5. Find the rank of $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (3)

6. Find the rank of the matrix, by reducing it to the normal form.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$ (3)

7. Find the Rank of the matrix $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ (3)

8. Find the Rank of the matrix, by reducing it to the normal form $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ (3)

9. Find the Rank of the matrix, by reducing it to the normal form. $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$ (3)

(10). Find the Rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$ by reducing it to the normal form. (3)

(11). $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ (4)

(12). $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ (3)

(13). $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & -2 \end{bmatrix}$ (4)

(16). $A = \begin{bmatrix} 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ (3)

(14). $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ (3)

(17). $A = \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$ (2)

(15). $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ (3)

(18). Find the Rank of the matrix, by reducing it to the normal form. (4)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$

$R_2 = R_2 - 4R_1$; $R_3 = R_3 - 3R_1$; $R_4 = R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

$R_4 = 2R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix}$$

$R_4 = \frac{R_4}{2}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_2 = C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_2 = \frac{C_2}{-7}$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$R_3 = R_3/_{-}$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 6 & -11 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_3 = C_3 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 6 & -11 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_4 = C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & -11 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_3 = C_3 - 6C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_4 = C_4 + 11C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_4 = C_4 - 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

$C_3 = \frac{C_3}{-1}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k-1 \end{bmatrix}$$

let $k-1=0 \Rightarrow \boxed{k=1}$, then the matrix will be of the form

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$.

$\therefore \boxed{\rho(A) = 3}$

Inverse of the matrix by elementary transformations (42):

(Gauss-Jordan method)

Suppose 'A' is a non-singular square matrix of order 'n'. we write $\boxed{A = I_n A}$ then we apply the elementary row operations only to the matrix 'A' and the Prefactor 'I_n' of R.H.S. we get an eqn of the form $\boxed{I_n = BA}$ then 'B' is an inverse of 'A'.

i.e, $\boxed{B = A^{-1}}$.

Problems :- ①. Find the inverse of the matrix 'A' using elementary operations where $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

we can write $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 = 2R_3 - R_2$ we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

$R_1 = R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

$R_1 = R_1 + 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

$R_2 = R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

$R_2 = \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & +10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A$$

this is in the form $I_3 = BA$

$$\therefore B = A^{-1} = \begin{bmatrix} 1 & -8 & +10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

②. $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: we can write $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 = R_1 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 = R_1 + R_3.$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 = \frac{R_2}{2}$; $R_3 = \frac{R_3}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} A$$

this is in the form of $I_3 = BA$.

$$\therefore B = A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

H.W (3). $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Sol: We can write $A = I_3 A$.

(4) $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

Sol:- We can write $A = I_4 A$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 = R_3 - 2R_1$; $R_4 = R_4 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_4 = R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_4 = 2R_4 - 3R_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

$R_1 = R_1 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

$R_1 = R_1 + R_4$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

$R_2 = R_2 + R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

$R_2 = R_2 - R_4$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

$R_3 = R_3 - 3R_4$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ 6 & -8 & 10 & -6 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

i. $R_3 = \frac{R_3}{-2}$; $R_4 = \frac{R_4}{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 2 & -2 \\ -3 & +4 & -5 & +3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

this is of the form $I_4 = BA$

i.e, $B = A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 2 & -2 \\ -3 & +4 & -5 & +3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

5. $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

6. $B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

7. $C = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

-: System of Linear Simultaneous Equations :-

An Eq^n of the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$

where $x_1, x_2, x_3, x_4, \cdots, x_n$ are unknowns and $a_1, a_2, a_3, \cdots a_n$ are constants is called a linear Eq^n in n-unknowns.

the system of Eq^n's can be written in the form

$\boxed{AX=B}$ — (1). where $A = [a_{ij}]_{m\times n}$; $X = (x_1, x_2, x_3, \cdots x_n)^T$

$B = (b_1, b_2, b_3, \cdots b_m)^T$

the matrix $[A|B]$ is called the "augumented matrix" of the system.

—* If $B=0$ in (1), then the system is said to be "homogenous". otherwise the system is said to be "non-homogenous".

—* the system $\boxed{AX=0}$ is always consistent, since $X=0$ i.e. $(x_1=0, x_2=0, x_3=0, \cdots, x_n=0)$ is always a sol^n of $AX=0$, this type of sol^n is called "trivial sol^n" of the system, otherwise the sol^n is called to be "non-trivial sol^n".

—* the system $\boxed{AX=B}$ is consistent i.e. it has a sol^n (unique or infinite) $\Leftrightarrow$ rank of A = rank $(A|B)$

—* If rank of $A \neq$ rank of $[A|B]$. then the system has no-sol^n.

—* If rank of A = rank of $[A|B] = r$ then the system consists unique sol^n.

* If rank of A = rank of [A|B] = r < number of unknowns then the system is consistent ∃ infinite no. of sol's.

—: Problems :—

① S.T $x-4y+7z=14$; $3x+8y-2z=13$; $7x-8y+26z=5$ are not consistent.

Sol: The given system of Eq's can be written as $\boxed{AX=B}$

where $A=\begin{bmatrix}1 & -4 & 7\\ 3 & 8 & -2\\ 7 & -8 & 26\end{bmatrix}$; $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$; $B=\begin{bmatrix}14\\ 13\\ 5\end{bmatrix}$

the Augmented matrix of the given Eq^n is

$$[A|B]=\begin{bmatrix}1 & -4 & 7 & 14\\ 3 & 8 & -2 & 13\\ 7 & -8 & 26 & 5\end{bmatrix}$$

$R_2=R_2-3R_1$; $R_3=R_3-7R_1$

$$[A|B]\sim\begin{bmatrix}1 & -4 & 7 & 14\\ 0 & 20 & -23 & -29\\ 0 & 20 & -23 & -93\end{bmatrix}$$

$R_3=R_3-R_2$

$$[A|B]\sim\begin{bmatrix}1 & -4 & 7 & 14\\ 0 & 20 & -23 & -29\\ 0 & 0 & 0 & -64\end{bmatrix}$$

Here the no. of non-zero rows $=3$ $\boxed{\therefore \rho[A|B]=3}$

& we can observe that the no. of non-zero rows in 'A' = '2'

$$\boxed{\therefore \rho(A) = 2}$$

$$\therefore \boxed{\rho(A) \neq \rho(A|B)}$$

$\therefore$ The system is inconsistent.

(2). S.T the Equations $x+y+z=4$; $2x+5y-2z=3$; $x+7y-7z=5$ are not consistent.

(3). show that the Equations $x-3y-8z=-10$; $3x+y-4z=0$; $2x+5y+6z=3$ are not consistent.

(4). S.T $x+2y+z=3$; $2x+3y+2z=5$; $3x-5y+5z=2$; $3x+9y-z=4$ are consistent and solve them.

A: the given system of Equations can be written as

$$\boxed{AX=B}$$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$

the Augumented matrix of the given Equation is

$$[A|B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$; $R_4 = R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix}$$

$R_4 = R_4 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

$R_4 = \frac{R_4}{-4}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 = R_3 - 11R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 = \frac{R_3}{2}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_4 = R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \sim (1)$$

This is an Echelon form and the no. of non-zero rows = 3.

$\boxed{\therefore \rho(A|B) = 3}$

& $\boxed{\rho(A) = 3}$

$\therefore \boxed{\rho(A) = \rho(A|B)}$

$\therefore$ the system is consistent.

[$\because$ from $\boxed{AX = B}$]

$$(1) \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

we get $x + 2y + z = 3$

$\Rightarrow x + 2 + 2 = 3$

$\Rightarrow \boxed{x = -1}$

$\Rightarrow -y = -1 \Rightarrow \boxed{y = 1}$

$\Rightarrow \boxed{z = 2}$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the solⁿ of the given system.

5. Find whether the following Equations are consistent, if so solve them $x+y+2z=4$; $2x-y+3z=9$; $3x-y-z=2$ (52)

Sol:- We write the given Equations in the form $\boxed{AX=B}$

where $A=\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 3\\ 3 & -1 & -1\end{bmatrix}$; $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$; $B=\begin{bmatrix}4\\ 9\\ 2\end{bmatrix}$

The Augmented matrix of the given Equation is

$$[A|B]=\begin{bmatrix}1 & 1 & 2 & 4\\ 2 & -1 & 3 & 9\\ 3 & -1 & -1 & 2\end{bmatrix}$$

Applying $R_2=R_2-2R_1$; $R_3=R_3-3R_1$, we get

$$[A|B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -1 & 1\\ 0 & -4 & -7 & -10\end{bmatrix}$$

$R_3=3R_3-4R_2$, we get

$$[A|B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -1 & 1\\ 0 & 0 & -17 & -34\end{bmatrix}$$

The matrix is Echelon form.

Here $\rho(A)=3$; $\rho(A|B)=3$.

$\boxed{\therefore \rho(A)=\rho[A|B]}$

The system of Equations is consistent.

Here the number of unknowns is '3'.

Since $\rho(A) = \rho[A|B]$ = no. of unknowns.

∴ the system of Equations has a unique solution.

we have
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$x+y+2z = 4$; $-3y-z=1$; $-17z=-34$

$\boxed{x=1}$. $\quad -3y = 1+2 \quad \boxed{z=2}$

$\boxed{y=-1}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$ is the sol^n of the given system.

6. Show that the Equations $x+y+z=6$; $x+2y+3z=14$; $x+4y+7z=30$ are consistent and solve them.

Sol:- We write the given Equations in the form $\boxed{AX=B}$

i.e,
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

the Augmented matrix $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

Applying $R_2 = R_2 - R_1$; $R_3 = R_3 - R_1$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

Applying $R_3 = R_3 - 3R_2$, we get

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form.

Here $\rho(A) = 2$; $\rho(A|B) = 2$.

Since $\rho(A) = \rho[A|B]$.

$\therefore$ The system of Eq^n's is Consistent.

Here the number of unknowns is '3'.

Since rank of 'A' is less than the number of unknowns.

$\therefore$ The system of Equations will have infinite no. of sol's interms of $n - r = 3 - 2 = 1$ (arbitrary constant)

$\therefore$ the given System of Equations reduced form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$x+y+z=6 \quad (1) \quad ; \quad y+2z=8 \quad (2)$

Let $\boxed{z=k}$

Put in (2) we get $\boxed{y=8-2k}$

put $z=k$; $y=8-2k$ in (1)

$(1) \Rightarrow x+8-2k+k=6 \Rightarrow \boxed{x=k-2}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k-2 \\ 8-2k \\ k \end{bmatrix}$$ where 'k' is an arbitrary constant.

(7). Discuss for what values of 'λ', 'μ' the simultaneous Eq^{n}s $x+y+z=6$; $x+2y+3z=10$; $x+2y+\lambda z=\mu$ have.

(i). no sol^n (ii). a unique sol^n (iii). Infinite many sol^{n}s.

Sol^n:- The matrix form of given system Eq^n is

$\boxed{AX=B.}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix is $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2 = R_2 - R_1$

$R_3 = R_3 - R_1$

$$\rightarrow \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\rightarrow \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix} \quad -(1)$$

Case (1) :- Let $\left.\begin{matrix}\lambda \neq 3 \\ \mu = 10\end{matrix}\right\}$ in (1) then $\rho(A) = 3$ & They have same Rank $\rho(A|B) = 3$

∴ the System is Consistent.

Here the no. of unknowns is '3', which is Same as the rank of 'A'

∴ $\rho(A) = \rho[A|B]$

∴ It has a unique sol^n.

Case (2) :- Let $\left.\begin{matrix}\lambda = 3 \\ \mu \neq 10\end{matrix}\right\}$ in (1) then $\rho(A) = 2$, $\rho[A|B] = 3$

∴ $\boxed{\rho(A) \neq \rho(A|B)}$

∴ the system of Equations has no sol^n.

Case (3) :- Let $\left.\begin{matrix}\lambda = 3 \\ \mu = 10\end{matrix}\right\}$ in (1) then $\rho(A) = \rho[A|B] = 2$.

∴ the given System of Eq^ns will be Consistent.

~~Here~~ But the no. of unknowns $= 3$

$\therefore r < n$

$2 < 3$

$\exists$ the system has infinitely many sol^{n}s.

$\boxed{\text{Note}}$: The System have a unique sol^n, we must have $\rho(A) = $ no. of unknowns. ie, $\det A \neq 0$.

---

(8). Find for what values of 'λ' the Eq^{n}s $x+y+z=1$; $x+2y+4z=\lambda$; $x+4y+10z=\lambda^2$ have a sol^n & solve them completely in each case

Sol^n:- The given System can be expressed as $AX=B$

ie, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$

The Augmented matrix $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$

Applying $R_2 = R_2 - R_1$; $R_3 = R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{bmatrix}$$

Applying $R_3 = R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{bmatrix} \quad \text{———} \quad (I)$$

the Eq's will be consistent $\Leftrightarrow \lambda^2-3\lambda+2=0$

$$\lambda^2-2\lambda-\lambda+2=0$$
$$\lambda(\lambda-2)-(\lambda-2)=0$$
$$(\lambda-2)(\lambda-1)=0$$
$$\lambda=2,1.$$

case(1) :- If $\lambda=1$ in (I) then we have

$$[A|\tilde{B}] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{--- (1)}$$

Here $\rho(A|B)=2$
& $\rho(A)=2$ & The number of unknowns $=3\ (n)$
Here $r<n$

$\therefore\ \rho(A)=\rho(A|B)=2<$ no. of unknowns (3). $\exists$ the system of the Eq's will have infinite no. of sol's interms of $n-r=3-2=1$ (arbitrary constant).

the given system of Eq's reduced form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x+y+z=1$; $y+3z=0$; $\boxed{z=k}$

$\boxed{y=-3k}$

$x=1-k+3k$

$\boxed{x=1+2k}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2k \\ -3k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$ where 'k' is a parameter.

Case (2) :- If $\lambda = 2$ in (A) then we have.

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{——} \ (2)$$

Here $\rho(A|B) = 2$

$\rho(A) = 2$.

Here $\rho(A|B) = \rho(A) = 2 <$ no. of unknowns (3) $\exists$ a system will have infinite no. of many sol's.

$$(2) \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow x+y+z=1$; $y+3z=1$; $\boxed{z=k}$

$x = 1-1+3k-k$; $\boxed{y = 1-3k}$

$\boxed{x = 2k}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ 1-3k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where 'k' is a parameter.

HW

9. Test for consistency and if consistent solve the system,

$5x+3y+7z=4$; $3x+26y+2z=9$; $7x+2y+10z=5$.

(or).

S.T the system of Eq's $5x+3y+7z=4$; $3x+26y+2z=9$; $7x+2y+10z=5$ is consistent & hence solve it.

(60)

(10) S.T the Eq^n $3x+4y+5z=a$; $4x+5y+6z=b$ & $5x+6y+7z=c$ donot have a sol^n unless $a+c=2b$.

Sol^n:- the given system is of the form $\boxed{AX=B}$.

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix $[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$

$R_2 = 3R_2 - 4R_1$; $R_3 = 3R_3 - 5R_1$

$$[A|B] \sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$[A|B] \sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3a-6b+3c \end{bmatrix}$$

From the matrix we can have.

$$3a+3c = 6b$$

$$\Rightarrow \boxed{a+c=2b}$$

//

11. Find the values of 'a' and 'b' for which the Eq's $x+ay+z=3$; $x+2y+2z=b$; $x+5y+3z=9$ are consistent. when will these Eq's have a unique sol?

Sol:- The matrix form of given system of Eq's are

$$\boxed{AX=B}$$

where $A=\begin{bmatrix}1 & a & 1\\1 & 2 & 2\\1 & 5 & 3\end{bmatrix}$; $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$; $B=\begin{bmatrix}3\\b\\9\end{bmatrix}$

we have the Augmented matrix is $[A|B]=\begin{bmatrix}1 & a & 1 & 3\\1 & 2 & 2 & b\\1 & 5 & 3 & 9\end{bmatrix}$

$R_2=R_2-R_1$; $R_3=R_3-R_1$

$$(A|B)\sim\begin{bmatrix}1 & a & 1 & 3\\0 & 2-a & 1 & b-3\\0 & 5-a & 2 & 6\end{bmatrix} \quad \text{——} \quad (1)$$

Since the system have a unique sol, we must have $\rho(A)$ = no. of unknowns i.e., $\det A\neq 0$.

$$|A|\neq 0 \Rightarrow \begin{vmatrix}1 & a & 1\\0 & 2-a & 1\\0 & 5-a & 2\end{vmatrix}\neq 0 \Rightarrow (2-a)2-(5-a)\neq 0$$

$$4-2a-5+a\neq 0$$

$$-a-1\neq 0 \Rightarrow \boxed{a\neq -1}$$

then if $a\neq -1$, we get the unique sol.

If $a=-1$ [$\because$ from (1)] we have

$$(A|B)\sim\begin{bmatrix}1 & -1 & 1 & 3\\0 & 3 & 1 & b-3\\0 & 6 & 2 & 6\end{bmatrix}$$

$R_3 = \frac{R_3}{2}$

$$[A|B] \sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 3 & 1 & 3 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$(A|B) \sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 0 & 0 & 6-b \end{bmatrix} \quad \text{———} \quad (2)$$

The Eq^ns will be consistent $\Leftrightarrow$ $6-b=0$

$\boxed{b=6}$.

Put $b=6$ in (2)

$$(A|B) \sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then $\rho(A|B) = 2$; $\rho(A) = 2$.

$\boxed{\therefore \rho(A|B) = \rho(A)}$

$\therefore$ It is Consistent.

$$\boxed{\begin{matrix} a = -1 \\ b = 6 \end{matrix}}$$

and If $\boxed{a \neq -1}$, then the system will be Consistent.

(12). S.T the Eq^{n}s $x+2y-z=3$; $3x-y+2z=1$; $2x-2y+3z=2$; $x-y+z=-1$ are consistent and solve them.

(13). Solve the system of Eq^{n}s $x+2y+3z=1$; $2x+3y+8z=2$; $x+y+z=3$.

(14). Test the System $x+2y-5z=-9$; $3x-y+2z=5$; $2x+3y-z=3$, $4x-5y+z=-3$ is Consistent (or) not & solve it.

(15). Test the System $x+y+z=1$; $x-y+2z=1$; $x-y+2z=5$; $2x-2y+3z=1$ is Consistent (or) not & solve it.

(16). Test the System $x+2y-5z=-9$; $3x-y+2z=5$; $2x+3y-z=3$, $4x-5y+z=-3$ is Consistent (or) not & solve it.

(17). Test for Consistency and if consistent solve the System, $5x+3y+7z=4$; $3x+26y+2z=9$; $3x+26y+2z=19$.

(18). Solve the System of linear Eq^{n}s by matrix method.

$x+y+z=6$; $2x+3y-2z=2$; $5x+y+2z=13$.

(19). Find the values of 'a' and 'b' for which the Eq^{n}s, $x+y+z=3$; $x+2y+2z=6$; $x+9y+az=b$ have

(i). No Sol^n (ii). A unique Sol^n (iii). Infinite no. of Sol^{n}s.

Consistency of System of Homogenous Linear Eqⁿs :-

Consider a system of 'm' Homogenous linear Eqⁿs in 'n'-unknowns, namely

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0 \\ \cdots \\ \cdots \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \leftarrow (1)$$

(1) can be written as $\boxed{AX=0}$, which is the matrix Eqⁿ.

Here 'A' is called Coefficient matrix.
'X' is called Variable matrix.
'O' is called Constant matrix.

It is clear that $x_1=0, x_2=0, \cdots x_n=0$ is a solⁿ of (1).
this is called "trivial solⁿ" of $\boxed{AX=0}$.
this $AX=0$ is always consistent i.e, it has a solⁿ
the trivial solⁿ is called the "Zero solⁿ".

Working Rule for finding the solⁿs of the Eqⁿ $AX=0$:-

— * If 'A' is a non-singular matrix (i.e, $\det A \neq 0$) then the linear system $AX=0$ has only the zero solⁿ.

— * the system $AX=0$ possesses a non-zero solⁿ $\Leftrightarrow$ 'A' is a singular matrix. ($\det A = 0$).

Let rank of $A = r$

number of unknowns $= n$. then

—* If $r = n \Rightarrow$ the system of Eq^n's have only trivial sol^n (i.e., zero sol^n).

—* If $r < n \Rightarrow$ the system of Eq^n's have an infinite no. of non-trivial sol^n. we shall have "$\underline{n-r}$" linearly independent sol^{n}s.

—: Problems :—

(1). Show that only real number 'λ' for which the system $x+2y+3z=\lambda x$; $3x+y+2z=\lambda y$; $2x+3y+z=\lambda z$ has non-zero sol^n is '6'. & solve them, when $\lambda=6$.

Sl^n:- Given system can be expressed as $\boxed{AX=O}$, where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \& \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the no. of variables $\boxed{n=3}$ (unknowns)

the given system of Eq^n's possess a non-zero sol^n (non-trivial sol^n) $\Leftrightarrow$ 'A' is a singular matrix.

i.e., $|A| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0.$$

$R_1 = R_1 + R_2 + R_3$.

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$C_2 = C_2 - C_1 \quad ; \quad C_3 = C_3 - C_1$

$$\Rightarrow 6-\lambda \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[(-2-\lambda)(-1-\lambda)+1\right] = 0$$

$\Rightarrow \quad 6-\lambda = 0$

$\boxed{\lambda = 6}$

$(2+\lambda)(1+\lambda)+1 = 0.$

$\Rightarrow 2+2\lambda+\lambda+\lambda^2+1 = 0$

$\Rightarrow \lambda^2+3\lambda+3 = 0$

$$\lambda = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{-3}}{2} = \frac{-3 \pm i\sqrt{3}}{2}$$

$\therefore \lambda = 6$ is the real value.

remaining are complex values.

when $\boxed{\lambda = 6}$ $\Rightarrow$ the given system becomes

$$\underset{A}{\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{O}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$R_2 = 5R_2 + 3R_1$; $R_3 = 5R_3 + 2R_1$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $\rho(A) = 2 \Rightarrow \boxed{r=2}$

$\boxed{n=3}$

$\therefore r < n$
$2 < 3$ $\exists$ infinite sol's exist.

& $n - r = 3 - 2 = 1$ (arbitrary constant)

$\Rightarrow -5x + 2y + 3z = 0$; $-19y + 19z = 0$; $\boxed{z = k}$

$-5x = -3k - 2k$ $\qquad \boxed{y = k}$

$\boxed{x = k}$.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ is the sol^n of the given System.

(2). Solve the system of Eq^{n}s

$x + y - 3z + 2w = 0$; $2x - y + 2z - 3w = 0$; $3x - 2y + z - 4w = 0$;

$-4x + y - 3z + w = 0$.

Sol^n:- The given system can be written as $\boxed{AX = 0}$.

where $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ & $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Consider $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$; $R_4 = R_4 + 4R_1$

$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$

$R_3 = \frac{R_3}{-5}$ we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix}$

$R_3 = 3R_3 + R_2$; $R_4 = 3R_4 + 5R_2$

$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -5 & -8 \end{bmatrix}$

$R_4 = 2R_4 + 5R_3$

$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -21 \end{bmatrix}$

this is in Echelon form.

$\boxed{\rho(A) = 4}$ i.e. $\boxed{r = 4}$

$\boxed{n = 4}$

$\therefore \boxed{r = n}$

$\therefore$ the system of Eq^n's have only trivial sol^n (zero sol^n)

$\therefore \boxed{x = y = z = w = 0}$ is the only sol^n.

(3). Solve the system of the Eq^n's

$$4x+2y+z+3w=0 \quad ; \quad 6x+3y+4z+7w=0 \quad ; \quad 2x+y+w=0.$$

sl^n:- the given system can be written as $\boxed{AX=O}$.

where $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ & $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Consider $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$R_2 = 4R_2 - 6R_1$; $R_3 = 2R_3 - R_1$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_2 = R_2/10$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = \frac{R_1}{4}$

$$\sim \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $\rho(A) = 2$ i.e., $\boxed{r=2}$ $\boxed{n=4}$

$\therefore r < n$

$2 < 4$ $\exists$ infinite no. of sol^{n}s exists.

& $\boxed{n-r = 4-2 = 2}$ (we get two arbitrary constants)

the given system can be reduced into

$$A \quad X = O$$

$$\begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + \frac{y}{2} + \frac{z}{4} + \frac{3w}{4} = 0$; $z + w = 0$; Let $\boxed{x = k_1}$

Let $\boxed{w = k_2}$

$z = -w$

$\boxed{z = -k_2}$

$\frac{y}{2} = -\frac{3w}{4} - \frac{z}{4} - x$

$\frac{y}{2} = \frac{-3k_2}{4} + \frac{k_2}{4} - k_1$

$\frac{y}{\cancel{2}} = \frac{-3k_2 + k_2 - 4k_1}{\cancel{4}\,2}$

$y = \frac{-2k_2 - 4k_1}{2}$

$\boxed{y = -(k_2 + 2k_1)}$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} k_1 \\ -2k_1 - k_2 \\ -k_2 \\ k_2 \end{bmatrix}$$ is the sol^n of the given system.

(4). S.T the system of Eq^{n}s $2x_1 - 2x_2 + x_3 = \lambda x_1$
$2x_1 - 3x_2 + 2x_3 = \lambda x_2$
$-x_1 + 2x_2 = \lambda x_3$ can possess a non-trivial sol^n only if $\lambda = 1$; $\lambda = -3$. obtain the general sol^n in each case.

Sl^n:- the given Eq's can be written as $\boxed{AX=0}$

Here $A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & (-3-\lambda) & 2 \\ -1 & 2 & -\lambda \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

the given system possess a non-zero sol^n (non-trivial sol^n)

$\Leftrightarrow$ $|A| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0.$$

$C_1 = C_1 + C_2 + C_3$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -2 & 1 \\ 1-\lambda & -3-\lambda & 2 \\ 1-\lambda & 2 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3-\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} = 0$$

$R_2 = R_2 - R_1$
$R_3 = R_3 - R_1$

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$$\Rightarrow (1-\lambda)\begin{vmatrix} 1 & -2 & 1 \\ 0 & -\lambda-1 & 1 \\ 0 & 4 & -\lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(-\lambda-1)^2 - 4\right] = 0 \Rightarrow 1-\lambda = 0 \quad \boxed{\lambda=1}$$

$$(-\lambda-1)^2 - 4 = 0$$
$$(\lambda+1)^2 = 4$$
$$\lambda^2 = 4-1-2\lambda$$
$$\lambda^2+2\lambda-3=0$$
$$\boxed{\lambda=1};\ \boxed{\lambda=-3}$$

If $\lambda=1$ in 'A'

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \quad R_2 = R_2 - 2R_1 \quad R_3 = R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ --- (1)}$$

Here $\boxed{\rho(A)=1}$ i.e. $\boxed{r=1}$ $\boxed{n=3}$

$r<n$

$1<3$ $\exists$ infinite soln exist

$\Rightarrow \boxed{n-r = 3-1=2}$ (arbitrary constants)

from (1), the matrix reduced form is $\boxed{AX=0}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 + x_3 = 0$; Let $\boxed{x_2 = k_1, \ x_3 = k_2}$

$x_1 = \boxed{-k_2 + 2k_1}$

If $\lambda=-3$ in 'A'

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix} \quad R_2 = 5R_2 - 2R_1 \quad R_3 = 5R_3 + R_1$$

$$\sim \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & 8 \\ 0 & 8 & 16 \end{bmatrix} \quad R_2 = \frac{R_2}{4} \quad R_3 = \frac{R_3}{8}$$

$$\sim \begin{bmatrix} 5 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 = R_3 - R_2$$

$$\sim \begin{bmatrix} 5 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ --- (2)}$$

Here $\boxed{r=2;\ n=3}$

$r<n$

$2<3$ $\exists$ infinite soln exist.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

is the sol^n of the given system.

$\Rightarrow \boxed{n - r = 3 - 2 = 1}$ (arbitrary constant)

from (2), the matrix reduced into

$\boxed{AX = 0}$

$$(2) \Rightarrow \begin{bmatrix} 5 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_1 - 2x_2 + x_3 = 0$; $x_2 + 2x_3 = 0$; $\boxed{x_3 = k_1}$

$5x_1 = -k_1 + 2(-2k_1)$ $\boxed{x_2 = -2k_1}$

$5x_1 = -5k_1$

$\boxed{x_1 = -k_1}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -2k_1 \\ k_1 \end{bmatrix}$$ is the sol^n of the given system.

(5). Solve $x + 3y - 2z = 0$; $2x - y + 4z = 0$; $x - 11y + 14z = 0$.

(6). Solve $3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$

$x + 3y + 13z + 3w = 0$.

(7). Determine 'b' such that the system of Homogenous Eq^{n}s $2x + y + 2z = 0$; $x + y + 3z = 0$; $4x + 3y + bz = 0$ has trivial and non-trivial sol^{n}s. Find the non-trivial sol^n.

(8). Determine the values of 'λ' for which the following set of Eq^{n}s may possess non-trivial sol^n.

$3x_1 + x_2 - \lambda x_3 = 0$; $4x_1 - 2x_2 - 3x_3 = 0$; $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$ & find general sol^n ...

(9). Examine whether the vectors are linearly dependent (74)
or not $(3,1,1), (2,0,-1), (4,2,1)$.

Sol: We can write the given vectors into.

$$3x+2y+4z=0 \quad \text{---} (1)$$
$$x+2z=0 \quad \text{---} (2)$$
$$x-y+z=0 \quad \text{---} (3).$$

Here $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ $\quad R_2 = 3R_2 - R_1 \; ; \; R_3 = 3R_3 - R_1$

$$\sim \begin{bmatrix} 3 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & -5 & -1 \end{bmatrix} \quad R_3 = 2R_3 - 5R_2$$

$$\sim \begin{bmatrix} 3 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & -12 \end{bmatrix}$$

Here this is in Echelon form.

$\boxed{\rho(A)=3}$

& $\boxed{n=3}$

$\therefore \boxed{r=n}$

We get the trivial soln exist.

$\boxed{x=y=z=0}$

$\therefore$ These vectors are linearly Independent.

Another method :—

If $\boxed{|A| \neq 0}$ then the vectors are "L.I" and if $\boxed{|A|=0}$ then the vectors are "L.D"

$$|A| = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$= 3(2) - 2(1-2) + 4(-1)$

$= 6+2-4 = 8-4$

$= 4 \neq 0$

$\therefore |A| \neq 0.$

$\therefore$ The vectors are L.I

(10). Determine whether the vectors $(1,2,3), (2,3,4), (3,4,5)$ are Linearly dependent (or) not.

(11). Find the values of 'α' such that the vectors $(1,1,0), (1,\alpha,0)$, and $(1,1,1)$ are L.I

* **Linear dependent of vectors (L.D)** :— V(F) be a vectorspace over a field 'F' then a finite set $\{\alpha_1, \alpha_2, ---, \alpha_n\}$ of vectors of 'V' is said to be L.D, if $\exists$ a scalar $\boxed{a_1, a_2, a_3, ---, a_n \neq 0}$ (not all zeros) $\ni a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + --- + a_n\alpha_n = 0$ then $\alpha_1, \alpha_2, ---, \alpha_n$ are called "L.D" of vectors.

* **Linear Independent of vectors (L.I)** :— V(F) be a vectorspace over a field 'F' then a finite set $\{\alpha_1, \alpha_2, --- \alpha_n\}$ of vectors of 'V' is said to be L.I, if $a_1\alpha_1 + a_2\alpha_2 + --- + a_n\alpha_n = 0$, where $a_1, a_2, a_3, ---, a_n \in F$ $\Rightarrow \boxed{a_1 = 0 ;\ a_2 = 0 ;\ a_3 = 0 \ ---\ ; a_n = 0}$ then $\alpha_1, \alpha_2, \alpha_3, ---- \alpha_n$ are called "L.I" of vectors.

(10). Determine whether the vectors (1, 2, 3), (2, 3, 4), (3, 4, 5) are L.D (or) not.

Sol:- we can write the given vectors into

$$x + 2y + 3z = 0 \quad \text{--- (1)}$$
$$2x + 3y + 4z = 0 \quad \text{--- (2)}$$
$$3x + 4y + 5z = 0 \quad \text{--- (3)}.$$

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$

(or)

Another method :-

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$|A| = 0$

$\therefore$ the vectors are L.D.

$$A \sim \begin{bmatrix} 3 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$A \sim \begin{bmatrix} 3 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here this is in Echelon form.

$\boxed{\rho(A)=2}$; $\boxed{n=3}$.

$\therefore \boxed{\rho(A) = r = 2}$

Here $\left.\begin{matrix} r<n \\ 2<3 \end{matrix}\right\} \exists$ infinite many sol's exist.

$\therefore \exists$ infinite many non-trivial sol's exist.

$\therefore$ these vectors are L.D. ($\because$ From definition)

(41). Find the values of 'α' suchthat the vectors $(1,1,0)$, $(1,\alpha,0)$, and $(1,1,1)$ are L.D.

Sol: We can write the given vectors into

$x+y+z=0$ — (1)

$x+\alpha y+z=0$ — (2)

$z=0$ — (3)

Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 = R_2 - R_1$

Another method :

(or) $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$ $\because$ Given vectors are L.D i.e., $|A|=0$

$= (\alpha - 1) = 0$

$\boxed{\alpha = 1}$

$$\therefore A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $\alpha - 1 = 0 \Rightarrow \boxed{\alpha = 1}$ then we get ($\because$ Given vectors are L.D)

$\boxed{\rho(A) = 2 = r}$ & $\boxed{n = 3}$

$\therefore \left.\begin{matrix} r < n \\ 2 < 3 \end{matrix}\right\}$ $\exists$ infinite many non-trivial sol^n exist.

Hence the vectors are L.D. ($\because$ From definition)

Gauss - Elimination method :—

The method of solving a system of n-linear Eq's in n-unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system, which may be solved by backward substitution. We discuss the method here for $n = 3$. The method is analogous for $n > 3$.

— Problems :—

①. Solve the Eq^{n}s $3x + y + 2z = 3$; $2x - 3y - z = -3$; $x + 2y + z = 4$ using Gauss-Elimination method.

Sol^n: The given system of the Eq^{n}s can be written in the matrix form is $\boxed{AX = B}$ $\Rightarrow$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

The Augmented matrix of the given System is

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$R_3 = 7R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

the Augmented matrix corresponds to the following uppertriangular system (by using the backward substitution)

$x + 2y + z = 4$; $-7y - 3z = -11$; $8z = -8$

$x = 4 - 2y - z$; $-7y = -11 + 3z$; $\boxed{z = -1}$

$\boxed{y = 2}$

$\boxed{x = 1}$

$\therefore$ the sol^n is $\boxed{\begin{array}{l} x = 1 \\ y = 2 \\ z = -1 \end{array}}$

(2). Solve the System of Equations $2x_1+x_2+2x_3+x_4=6$

$6x_1-6x_2+6x_3+12x_4=36$; $4x_1+3x_2+3x_3-3x_4=-1$; $2x_1+2x_2-x_3+x_4=10$.

using Gauss - elimination method.

Sol: the given system of Eq^n can be written in the matrix form as $\boxed{AX=B}$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

the Augmented matrix of the given system is

$$[A|B] \sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$R_2 = \frac{R_2}{6}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$; $R_3 = R_3 - 4R_1$; $R_4 = R_4 - 2R_1$

$$[A|B] \sim \left[\begin{array}{ccccc} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{array}\right]$$

$R_3 = 3R_3 - 7R_2$; $R_4 = 3R_4 - 4R_2$

$$\sim \left[\begin{array}{ccccc} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & -9 & 3 & 18 \end{array}\right]$$

$R_4 = R_4 - 3R_3$

$$\sim \left[\begin{array}{ccccc} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 0 & 39 & 117 \end{array}\right]$$

this corresponds to the upper triangular System is

$x_1 - x_2 + x_3 + 2x_4 = 6$; $3x_2 - 3x_4 = -6$; $-3x_3 - 12x_4 = -33$; $39x_4 = 117$

$\boxed{x_1 = 2}$ $\boxed{x_2 = 1}$ $\boxed{x_3 = -1}$ $\boxed{x_4 = 3}$

$\therefore$ the soln is $\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 3 \end{array}}$

(3). use the Gauss Elimination method to solve

$x + 2y - 3z = 9$; $2x - y + z = 0$; $4x - y + z = 4$.

Gauss - Seidel Iteration method :—

(1). Use Gauss-Seidel iteration method to solve the system.

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14.$$

Sol: the given system is diagonally dominant and we write it as

$$x = \frac{1}{10}\left[12-y-z\right] \quad \text{———} \quad (1)$$
$$y = \frac{1}{10}\left[13-2x-z\right] \quad \text{———} \quad (2)$$
$$z = \frac{1}{10}\left[14-2x-2y\right] \quad \text{———} \quad (3)$$

Iteration (1) :—

→ we start iteration by taking $y=0$; $z=0$ in (1), we get

$$\boxed{x^{(1)} = 1.2}$$

Put $x=x^{(1)}=1.2$; $z=0$ in (2), we get

$$\boxed{y^{(1)} = 1.06}$$

Put $y^{(1)}=1.06$; $x^{(1)}=1.2$ in (3), we get

$$\boxed{z^{(1)} = 0.95}$$

Iteration (2) :— Now taking $y=y^{(1)}$; $z=z^{(1)}$ in (1), we get

$$\boxed{x^{(2)} = 0.999}$$

Put $x=x^{(2)}=0.999$ & $z=z^{(1)}$ in (2), we get

$$\boxed{y^{(2)} = 1.005}$$

Put $x=x^{(2)}$; $y=y^{(2)}$ in (3), we get

$$\boxed{z^{(2)} = 0.999}$$

Iteration (3) :-

Again taking $y = y^{(2)}$; $z = z^{(2)}$ in (1), we get

$$\boxed{x^{(3)} = 1.00}$$

Put $x = x^{(3)}$; $z = z^{(2)}$ in (2), we get

$$\boxed{y^{(3)} = 1.00}$$

Put $x = x^{(3)}$; $y = y^{(3)}$ in (3), we get

$$\boxed{z^{(3)} = 1.00}$$

Iteration (4) :- Again taking $y = y^{(3)}$; $z = z^{(3)}$ in (1), we get

$$\boxed{x^{(4)} = 1.00}$$

Put $x = x^{(4)}$; $z = z^{(3)}$ in (2), we get

$$\boxed{y^{(4)} = 1.00}$$

Put $x = x^{(4)}$; $y = y^{(4)}$ in (3) ; we get

$$\boxed{z^{(4)} = 1.00}$$

we tabulate the results as follows :

Variable	I st approx.	II nd approx.	III rd approx.	IV th approx
x	1.20	0.999	1.00	1.00
y	1.06	1.005	1.00	1.00
z	0.95	0.999	1.00	1.00.

thus the soln of the given system of the Equations is

$$\boxed{x = 1 \; ; \; y = 1 \; ; \; z = 1}.$$

(2). Solve the following System of Equations by Gauss-Seidel method.

$$8x_1 - 3x_2 + 2x_3 = 20$$

$$4x_1 + 11x_2 - x_3 = 33$$

$$6x_1 + 3x_2 + 12x_3 = 36$$

Sol:- The given System is diagonally dominant and we write it as

$$x_1 = \frac{1}{8}(20 + 3x_2 - 2x_3) \quad \text{——— (1)}$$

$$x_2 = \frac{1}{11}(33 - 4x_1 + x_3) \quad \text{——— (2)}$$

$$x_3 = \frac{1}{12}(36 - 6x_1 - 3x_2) \quad \text{——— (3)}$$

1st approximations :— put $x_2 = 0$; $x_3 = 0$ in (1) ; we get

$$x_1^{(1)} = \frac{1}{8}\left[20 + 3(0) - 2(0)\right] = 2.5$$

Put $x_1 = x_1^{(1)}$; $x_3 = 0$ in (2) ; we get

$$x_2^{(1)} = \frac{1}{11}\left[33 - 4x_1^{(1)} + 0\right] = 2.1$$

Put $x_1 = x_1^{(1)}$; $x_2 = x_2^{(1)}$ in (3) ; we get

$$x_3^{(1)} = \frac{1}{12}\left[36 - 6x_1^{(1)} - 3x_2^{(1)}\right] = 1.2$$

2nd approximations :-

$$x_1^{(2)} = \frac{1}{8}\left[20 + 3x_2^{(1)} - 2x_3^{(1)}\right] = 2.988$$

$$x_2^{(2)} = 1/11\left[33 - 4x_1^{(2)} + x_3^{(1)}\right] = 2.023$$

$$x_3^{(2)} = 1/12\left[36 - 6x_1^{(2)} - 3x_2^{(2)}\right] = 1.00$$

3rd approximations :—

$$x_1^{(3)} = 1/8\left[20+3x_2^{(2)}-2x_3^{(2)}\right] = 3.0086$$

$$x_2^{(3)} = 1/11\left[33-4x_1^{(3)}+x_3^{(2)}\right] = 1.9969$$

$$x_3^{(3)} = 1/12\left[36-6x_1^{(3)}-3x_2^{(3)}\right] = 0.9965$$

4th approximations :—

$$x_1^{(4)} = 1/8\left[20+3x_2^{(3)}-2x_3^{(3)}\right]$$

$$x_2^{(4)} = 1/11\left[33-4x_1^{(4)}+x_3^{(3)}\right]$$

$$x_3^{(4)} = 1/12\left[36-6x_1^{(4)}-3x_2^{(4)}\right]$$

5th approximations :—

$$x_1^{(5)} = 1/8\left[20+3x_2^{(4)}-2x_3^{(4)}\right]$$

$$x_2^{(5)} = 1/11\left[33-4x_1^{(5)}+x_3^{(4)}\right]$$

$$x_3^{(5)} = 1/12\left[36-6x_1^{(5)}-3x_2^{(5)}\right]$$

we tabulate the results as follows :—

Proceeding like this, we get

variable	1th appn	2nd appn	3rd appn	4th appn	5th appn
x_1	2.5	2.988	3.0086	2.9997	2.9998
x_2	2.1	2.023	1.9969	1.9998	2.000
x_3	1.2	1.000	0.9965	1.0002	1.000

Thus the required sol^n is

$$\boxed{x_1 = 2.9998 \; ; \quad x_2 = 2.000 \; ; \quad x_3 = 1.000}$$

H.W

3. $x + 10y + z = 6 \; ; \; 10x + y + z = 6 \; ; \; x + y + 10z = 6.$

4. $10x - 2y - z - u = 3 \; ; \; -2x + 10y - z - u = 15$

$-x - y + 10z - 2u = 27 \; ; \; -x - y - 2z + 10u = -9.$

by using Gauss - Seidel method

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—* UNIT-II *—

—: Eigen values and Eigen Vectors :—